

ESSENTIAL PERTURBATIONS OF POLYNOMIAL VECTOR FIELDS WITH A PERIOD ANNULUS

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ABSTRACT. In this paper we first give the explicit definition of essential perturbation. Secondly, given a perturbation of a particular family of centers of polynomial differential systems of arbitrary degree for which we explicitly know its Poincaré–Liapunov constants, we give the structure of its k -th Melnikov function. This result generalizes the result obtained by Chicone and Jacobs for perturbations of degree at most two of any center of a quadratic polynomial system. Moreover we study the essential perturbations for all the centers of the differential systems

$$\dot{x} = -y + P_d(x, y), \quad \dot{y} = x + Q_d(x, y),$$

where P_d and Q_d are homogeneous polynomials of degree d , for $d = 2$ and $d = 3$.

1. INTRODUCTION

One of the last open problems from the list suggested by Hilbert at the beginning of the 20th century is the 16th problem, see [15]. The second part of this problem focus on the study of the limit cycles of planar polynomial real differential systems. More specifically Hilbert’s 16th problem (part b) is the following:

For the family of polynomial differential systems of degree d , is there a uniform upper bound, depending only on d , for the number of limit cycles of each system in the family?

This problem is still unsolved even for quadratic systems, i.e. for the case $d = 2$ (see [8]). Moreover, a weaker version is included in Smale’s list of problems to be solved for the 21st century (see [29]).

Roussarie establishes in [25] that this global problem can be reduced to several local bifurcation problems. In fact, finite cyclicity of any limit periodic set, in terms of the degree of the system, implies the solution to Hilbert’s 16th problem. The cyclicity problem has been studied by several authors also in its relation with the center problem, see for instance [4, 5, 7, 10, 11, 12, 18]. Roughly speaking, the cyclicity of a limit periodic set of a polynomial system of degree at most d , is the maximum number of limit cycles that can bifurcate from the given limit periodic set inside the family of all polynomial systems of degree d ; see Definition 12 in [25] for a precise definition. Some usual examples of limit periodic sets are: a weak focus, a center point, a period annulus, a homoclinic loop, a heteroclinic graph.

In this work we contribute to the study of the cyclicity of a period annulus \mathcal{P} surrounding a nondegenerate center point. In order to state our contribution we introduce some notations.

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Consider a (fixed) system with a nondegenerate center at the origin:

$$(1) \quad \dot{x} = -y + P(x, y), \quad \dot{y} = x + Q(x, y),$$

where P and Q are real polynomials of degree at most d without constant nor linear terms. For system (1), there exists an analytical first integral $H(x, y)$ and an inverse integrating factor $V(x, y)$ with $V(0, 0) = 1$, see [14, 22, 23]. The periodic orbits $\gamma_h \subset \{H = h\}$ surrounding the origin of (1) can be parameterized by the values of H . The period annulus \mathcal{P} is defined by

$$\mathcal{P} = \{\gamma_h : h \in (h_0, h_1)\},$$

where $h_0 \in \mathbb{R}$ corresponds to the inner boundary (i.e. the origin) and $h_1 \in \mathbb{R} \cup \{+\infty\}$ corresponds to the outer boundary. Consider now a family of perturbations of (1):

$$(2) \quad \begin{aligned} \dot{x} &= -y + P(x, y) + \varepsilon p(x, y, \tilde{\lambda}, \varepsilon), \\ \dot{y} &= x + Q(x, y) + \varepsilon q(x, y, \tilde{\lambda}, \varepsilon), \end{aligned}$$

where p and q are polynomials in x, y of degree d and analytic functions in the small *bifurcation parameter* ε and in the parameters $\tilde{\lambda} \in \mathbb{R}^m$. We remind that we consider the problem of bifurcation of limit cycles from the period annulus \mathcal{P} of system (1) in the family (2). The two mostly used methods to solve this problem are the averaging method, see for instance [1], and the Melnikov functions, see for instance [25, 26]. In this paper we mainly deal with the computation of the so called *Melnikov functions*. However, Melnikov functions cannot always be explicitly computed. In order to define what is a Melnikov function, we consider the Poincaré map $\pi(\cdot; \varepsilon) : \Sigma \rightarrow \Sigma$ associated to system (2) and the period annulus \mathcal{P} , where Σ is a transversal section parameterized by h , passing through the origin and cutting the whole \mathcal{P} . We are under the assumption that for $\varepsilon = 0$ system (2) has a center at the origin, thus we have that $\pi(h; 0) = h$ for all $h \in [h_0, h_1]$. By the analyticity of the Poincaré map with respect to parameters, we have the displacement map

$$d(h; \varepsilon) = \pi(h; \varepsilon) - h = M_1(h)\varepsilon + M_2(h)\varepsilon^2 + \dots + M_r(h)\varepsilon^r + \mathcal{O}(\varepsilon^{r+1}).$$

Depending on the parameters $\tilde{\lambda}$, there exists some $k \geq 1$ such that $M_r(h) \equiv 0$ for any $1 \leq r < k$ and $M_k(h) \neq 0$, i.e.

$$d(h; \varepsilon) = M_k(h)\varepsilon^k + \mathcal{O}(\varepsilon^{k+1}).$$

The function $M_k(h)$ is called the Melnikov function of order k . The isolated zeroes of $M_k(h)$ (counted with multiplicity) allow to study limit cycles of system (2) which bifurcate from the orbits of the period annulus of system (1) (see, for instance, subsection 4.3.4 of [25]). In particular, the following result, which is Theorem 6.1 in [19], is well-known.

Theorem 1. *Let $M_k(h)$ be the Melnikov function of order k associated to system (2) and let $h^* \in (h_0, h_1)$. We denote by $\gamma_h \subset \{H = h\}$ the periodic orbits surrounding the origin of (1). The following statements hold.*

- (i) *If there exists a limit cycle $\Gamma_{\varepsilon, h^*}$ of system (2) such that $\Gamma_{\varepsilon, h^*} \rightarrow \gamma_{h^*}$ as $\varepsilon \rightarrow 0$, then $M_k(h^*) = 0$.*

- (ii) If $M_k(h^*) = 0$ and $M'_k(h^*) \neq 0$, then there exists a hyperbolic limit cycle $\Gamma_{\varepsilon, h^*}$ of system (2) such that $\Gamma_{\varepsilon, h^*} \rightarrow \gamma_{h^*}$ as $\varepsilon \rightarrow 0$.
- (iii) If $M_k^{(i)}(h^*) = 0$ for $i = \overline{0, r-1}$ and $M_k^{(r)}(h^*) \neq 0$ (that is, h^* is a zero of multiplicity r of $M_k(h)$), then (2) has at most r limit cycles for ε sufficiently small in the vicinity of γ_{h^*} .
- (iv) The total number of isolated zeros of $M_k(h)$ (taking into account their multiplicity) is an upper bound for the number of limit cycles of system (2) that bifurcate from the periodic orbits of the considered period annulus of (1).

We have defined Melnikov functions in terms of the displacement map. Another way to compute the function $M_1(h)$ is through the following line integral:

$$M_1(h) = \oint_{H(x,y)=h} \frac{q(x,y,0) dx - p(x,y,0) dy}{V(x,y)},$$

where $V(x, y)$ is an inverse integrating factor of system (1) corresponding to the first integral H . The expression of the Melnikov function of order k involve, in general, iterated integrals up to order k (see again, for instance, subsection 4.3.4 of [25]). Hence, the explicit computation of high order Melnikov functions may become computationally cumbersome or even impossible. It turns out, however, that the expressions of the Melnikov functions for a particular example, obey some pattern. The aim of this work is to unveil this pattern. More exactly, we show in Theorem 4 that the Melnikov function of order k of system (2) is a (finite) linear combination of some functions, which we denote by $B_1(h)$, $B_3(h)$, \dots , $B_{2N+1}(h)$. These functions do not depend neither on k , nor on the parameters $\tilde{\lambda}$. The coefficients of the linear combination can be found only if one knows the expression of the Poincaré–Liapunov constants for the family. For certain particular cases, this has already been shown in [6, 16]. We give in Section 2 a general framework to this approach.

Chicone and Jacobs in [6] and Iliev in [16] succeeded to find the essential perturbations of quadratic systems when considering the problem of finding the cyclicity of a period annulus. We present in the sequel a formal definition of this notion.

Definition 2. *Given a parametric family of planar polynomial differential systems (2) which unfold a system with a period annulus \mathcal{P} , an essential perturbation is a choice of the parameters $\tilde{\lambda}$ such that:*

- (i) *the number of isolated zeros (counted with multiplicity) of the corresponding Melnikov function is greater or equal to the number of isolated zeros (counted with multiplicity) of the Melnikov function corresponding to any other value of $\tilde{\lambda}$;*
- (ii) *the order of the Melnikov function which satisfies (i) is the lowest possible;*
- (iii) *the number of involved parameters is the lowest possible satisfying (i) and (ii).*

In Section 2, we explain how to find the essential perturbations of a parametric family of planar polynomial differential systems which unfold a system with a period annulus surrounding a nondegenerate center by using Theorem 4.

Section 3 contains the description of the essential perturbations for all the centers of the form

$$\dot{x} = -y + P_d(x, y), \quad \dot{y} = x + Q_d(x, y),$$

where P_d and Q_d are homogeneous polynomials of degree d , for $d = 2$ and $d = 3$. We remark that the quadratic case was already described in [16] for quadratic systems written in complex form and we consider systems written in Bautin form. In Section 4 we correct the study of a particular quadratic system which appears in [3]. Last Section 5 contains a remark about the finiteness of the number of limit cycles bifurcating from the considered period annulus \mathcal{P} .

2. ESSENTIAL PERTURBATIONS. A GENERAL FRAMEWORK

First we consider a parametric family of planar polynomial real systems of the form

$$(3) \quad \dot{x} = -y + \lambda_1 x + P(x, y, \lambda), \quad \dot{y} = x + \lambda_1 y + Q(x, y, \lambda),$$

where P and Q are polynomials in x, y and $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n) \in \mathbb{R}^n$, and the subdegree in x and y of P and Q is at least 2. We consider a fixed $\lambda^* \in \mathbb{R}^n$ and we assume that for $\lambda = \lambda^*$ system (3) has a center at the origin, whose period annulus is denoted by \mathcal{P} . Like we discussed in the Introduction for systems (1) and (2), we consider a section Σ through the origin, transversal to the flow of (3) in the whole period annulus \mathcal{P} when λ is in a small neighborhood of λ^* , and parameterized by h . This time we assume that $h = 0$ corresponds to the origin of coordinates. We also consider the displacement map $d(h; \lambda) = \pi(h; \lambda) - h$ associated to family (3). We recall that $\pi(h; \lambda)$ denotes the Poincaré map, emphasizing that here it depends on the parameters λ .

The basic idea to tackle the bifurcation of limit cycles from \mathcal{P} is founded on properties of zeros of analytic functions of several variables depending on parameters. We will mainly use the description of these ideas given in subsection 6.1 of the book [24]. The same tools can be found in chapter 4 of the book of Roussarie [25], see also [26]. We denote by $v_{2j+1}(\lambda)$, $j = \overline{0, N}$, the Poincaré–Liapunov constants associated to the origin of the family of polynomial differential systems (3). See, for instance, chapter 3 in [24] for their definition. The Poincaré–Liapunov constants are the basic tool to solve the center problem, see e.g. [2, 5, 20, 27, 28]. We remark that the Poincaré–Liapunov constants are polynomials in λ and that their number $N + 1$ only depends on the considered family (3). The following statement corresponds to Lemma 6.1.6 in [24] but written with our notation and our assumptions.

Lemma 3. *There exist positive numbers ε_1 and δ_1 such that the displacement map $d(h; \lambda)$ is analytic for $|h| < \varepsilon_1$ and $\|\lambda - \lambda^*\| < \delta_1$ and there exist $N + 1$ analytic functions $b_{2j+1}(h, \lambda)$, $j = \overline{0, N}$, with $b_{2j+1}(0, \lambda^*)$ a nonzero constant, such that*

$$(4) \quad d(h; \lambda) = \sum_{j=0}^N v_{2j+1}(\lambda) h^{2j+1} b_{2j+1}(h, \lambda)$$

holds in the set $\{(h, \lambda) : |h| < \varepsilon_1 \text{ and } \|\lambda - \lambda^\| < \delta_1\}$.*

In addition, and due to the structure of the Poincaré–Liapunov constants, it is known that $v_1(\lambda) = \lambda_1$, $b_1(0, \lambda) = (e^{2\pi\lambda_1} - 1)/\lambda_1$, and, for each $j > 0$, $v_{2j+1}(\lambda)$ and $b_{2j+1}(h, \lambda)$ do not depend on λ_1 . We remark that the development of $d(h, \lambda)$ in powers of h given in (4) appears when a first integral $H(x, y)$ of system (3) with $\lambda = \lambda^*$ of the form $H(x, y) = \sqrt{x^2 + y^2} + o\left(\sqrt{x^2 + y^2}\right)$ is used. Some of the $v_{2j+1}(\lambda)$ might be identically null and in such a case we take, by default, the corresponding $b_{2j+1}(h, \lambda)$ constant equal to 1. We note that the Poincaré–Liapunov constants can be computed by algebraic methods, but the computations are usually cumbersome. Other useful remarks are that, since for $\lambda = \lambda^*$ system (3) has a center at the origin, we must have $d(h; \lambda^*) \equiv 0$ and, consequently,

$$v_{2j+1}(\lambda^*) = 0 \quad \text{for } j = \overline{0, N},$$

and that the functions $h^{2j+1}b_{2j+1}(h, \lambda)$ for $j = \overline{0, N}$ are linearly independent on a sufficiently small neighborhood of $(h, \lambda) = (0, \lambda^*)$.

We consider now a small bifurcation parameter ε and that, in (3), $\lambda = \lambda(\varepsilon)$ depends analytically on ε such that $\lambda(0) = \lambda^*$. We denote by $\lambda_{i,0}$ the i^{th} coordinate of the point $\lambda^* \in \mathbb{R}^n$, that is, $\lambda^* = (\lambda_{1,0}, \lambda_{2,0}, \dots, \lambda_{n,0})$. In addition, we take the series expansions

$$\lambda_i(\varepsilon) = \sum_{\ell \geq 0} \lambda_{i,\ell} \varepsilon^\ell,$$

for some reals $\lambda_{i,\ell}$. So now we see system (3), i.e.

$$(5) \quad \dot{x} = -y + \lambda_1(\varepsilon)x + P(x, y, \lambda(\varepsilon)), \quad \dot{y} = x + \lambda_1(\varepsilon)y + Q(x, y, \lambda(\varepsilon)),$$

as a one-parameter analytic perturbation of the period annulus surrounding the origin of system (3) when $\lambda = \lambda^*$. We emphasize that (5) depends on the parameters $\tilde{\lambda} = (\lambda_{i,\ell} : i = \overline{1, n}, \ell \geq 0)$. Hence (5) is like system (2) from the Introduction.

The displacement map of (5) is $d(h; \lambda(\varepsilon))$ and we assume that its Taylor series expansion in a neighborhood of $\varepsilon = 0$ takes the form

$$(6) \quad d(h; \lambda(\varepsilon)) = M_k(h) \varepsilon^k + \mathcal{O}(\varepsilon^{k+1}),$$

where $M_k(h)$ is the Melnikov function (of order $k \geq 1$). As we have remarked in the Introduction, $M_k(h)$ is in fact defined and analytic not only in a small neighborhood of $h = 0$, but on the whole Σ , i.e. on the interval $[0, h_1)$. The notations Σ and h_1 are given in the Introduction. This analyticity property is a consequence of the Global Bifurcation Lemma, referred as Lemma 2.2 in the work [6].

In order to present the main result of this Section, we need to identify the coefficients of the power series expansions of the Poincaré–Liapunov constants:

$$(7) \quad v_{2j+1}(\lambda(\varepsilon)) = \sum_{r \geq 1} \bar{v}_{2j+1,r} \varepsilon^r, \quad j = \overline{0, N}.$$

It can be shown that, for each $j = \overline{0, N}$, $\bar{v}_{2j+1,r}$ are polynomials in $(\lambda_{i,\ell} : i = \overline{1, n}, \ell = \overline{0, r})$.

Theorem 4. *There are $N + 1$ linearly independent functions $h^{2j+1}B_{2j+1}(h)$ which are analytic in $[0, h_1)$ and with $B_{2j+1}(0)$ a nonzero constant for $j = \overline{0, N}$, such that the Melnikov function of system (5) writes as*

$$(8) \quad M_k(h) = \sum_{j=0}^N \bar{v}_{2j+1,k} h^{2j+1} B_{2j+1}(h),$$

where $M_k(h)$ is defined in (6).

Proof. This proof is inspired by the one given by Chicone and Jacobs in [6] for the case that system (3) is a quadratic system written in Bautin normal form. We consider the functions $b_{2j+1}(h, \lambda)$, for $j = \overline{0, N}$, defined in (4) which are analytic in a neighborhood of $h = 0$. We define

$$B_{2j+1}(h) := b_{2j+1}(h, \lambda^*)$$

which is an analytic function in a neighborhood of $h = 0$ and verifies that $B_{2j+1}(0) = b_{2j+1}(0, \lambda^*)$ is a nonzero constant. Hence, we have that the $N + 1$ functions $h^{2j+1}B_{2j+1}(h)$ for $j = \overline{0, N}$ are linearly independent because each of them has a different subdegree in h . We have that

$$b_{2j+1}(h, \lambda(\varepsilon)) = B_{2j+1}(h) + \varepsilon R_{2j+1}(h, \varepsilon),$$

where $R_{2j+1}(h, \varepsilon)$ is an analytic function in a neighborhood of $(h, \varepsilon) = (0, 0)$. We substitute the latter expression of $b_{2j+1}(h, \lambda(\varepsilon))$ and the expansion of $v_{2j+1}(\lambda(\varepsilon))$ given in (7) in the expression (4) of the displacement map, that is

$$\begin{aligned} d(h; \lambda(\varepsilon)) &= \sum_{j=0}^N v_{2j+1}(\lambda(\varepsilon)) h^{2j+1} b_{2j+1}(h, \lambda(\varepsilon)) \\ &= \sum_{j=0}^N \left[\sum_{r \geq 1} \bar{v}_{2j+1,r} \varepsilon^r \right] h^{2j+1} \left(B_{2j+1}(h) + \varepsilon R_{2j+1}(h, \varepsilon) \right) \\ &= \sum_{r \geq 1} \sum_{j=0}^N (\bar{v}_{2j+1,r} h^{2j+1} B_{2j+1}(h)) \varepsilon^r + \bar{v}_{2j+1,r} h^{2j+1} R_{2j+1}(h, \varepsilon) \varepsilon^{r+1}. \end{aligned}$$

By (6) we are under the assumption that the lowest order term in the expansion of $d(h; \lambda(\varepsilon))$ in powers of ε corresponds to the power ε^k . If $k = 1$ we conclude that

$$M_1(h) = \sum_{j=0}^N \bar{v}_{2j+1,1} h^{2j+1} B_{2j+1}(h).$$

If $k > 1$, we deduce that, for $r = \overline{1, k-1}$, we have

$$\sum_{j=0}^N \bar{v}_{2j+1,r} h^{2j+1} B_{2j+1}(h) \equiv 0.$$

Since the functions $h^{2j+1}B_{2j+1}(h)$ for $j = \overline{0, N}$, are linearly independent, we deduce that $\bar{v}_{2j+1, r} = 0$ for $r = \overline{1, k-1}$ and $j = \overline{0, N}$. Therefore,

$$d(h; \lambda(\varepsilon)) = \left(\sum_{j=0}^N \bar{v}_{2j+1, k} h^{2j+1} B_{2j+1}(h) \right) \varepsilon^k + \mathcal{O}(\varepsilon^{k+1}).$$

Equating the coefficients of ε^k in this expression and in (6), we conclude that

$$M_k(h) = \sum_{j=0}^N \bar{v}_{2j+1, k} h^{2j+1} B_{2j+1}(h).$$

□

We refer to the $N + 1$ linearly independent functions $h^{2j+1}B_{2j+1}(h)$, for $j = \overline{0, N}$, as the *Bautin functions* associated to family (3). As a consequence of Theorem 4, we have that if one knows the Bautin functions, the study of the Melnikov function $M_k(h)$ reduces to the study of the coefficients $\bar{v}_{2j+1, k}$. We remark that since $v_{2j+1}(\lambda)$ are polynomials in λ , we have that $\bar{v}_{2j+1, k}$ are polynomials in $(\lambda_{i, \ell} : i = \overline{1, n}, \ell = \overline{1, k})$ and that there is a recursive way to give the expression of $\bar{v}_{2j+1, k}$ if k is high enough. In order to make this statement more precise, we make some notations. Let Λ_k be the real algebraic manifold

$$\Lambda_k := \{(\lambda_{i, \ell} : i = \overline{1, n}, \ell = \overline{1, k}) \mid M_r(h) \equiv 0, r = \overline{1, k-1}\} \subseteq \mathbb{R}^{nk}$$

and we consider the map $\phi_k : \Lambda_k \rightarrow \mathbb{R}^{N+1}$ given by

$$(9) \quad \phi_k : \Lambda_k \mapsto (\bar{v}_{2j+1, k} : j = \overline{0, N}).$$

It is important to study the range of the map ϕ_k for any $k \geq 1$. If it is possible to choose k^* to be the smallest k such that the range of the map ϕ_{k^*} is equal or contains the range of ϕ_k for any other k , then $M_{k^*}(h)$ will be the *essential Melnikov function* and k^* will be the *essential order*. After choosing k^* , we choose the *essential parameters*, which are a parametrization of a submanifold of Λ_{k^*} with the lowest possible dimension on which ϕ_{k^*} attains the maximal range. In fact, we fix the values of the *non-essential parameters* (most of them will be taken to be 0) in order that the expression of ϕ_{k^*} is the simplest possible but still maintains the maximal range.

3. ESSENTIAL PERTURBATIONS OF QUADRATIC AND CUBIC SYSTEMS

3.1. Essential perturbations of quadratic centers. In this paragraph we consider $d = 2$ and we will give our results for quadratic systems like (2) in the standard Bautin form. In fact, after an appropriate affine transformation (analytic with respect to ε), system (2) for $d = 2$ can be put into the standard Bautin form

$$(10) \quad \begin{aligned} \dot{x} &= \lambda_1 x - y - \lambda_3 x^2 + (2\lambda_2 + \lambda_5) xy + \lambda_6 y^2, \\ \dot{y} &= x + \lambda_1 y + \lambda_2 x^2 + (2\lambda_3 + \lambda_4) xy - \lambda_2 y^2, \end{aligned}$$

where the coefficients $\lambda(\varepsilon) = (\lambda_1(\varepsilon), \dots, \lambda_6(\varepsilon))$ are analytic functions for $|\varepsilon|$ sufficiently small and such that, for $\varepsilon = 0$ (i.e. for $\lambda(0) = (\lambda_{1,0}, \dots, \lambda_{6,0})$), system (10) has a center at the origin. We will apply Theorem 4 and other ideas presented in

Section 2. As it is proved in [2], see also [27] and references therein, there are four Poincaré–Liapunov constants of (10) (hence $N = 3$ in this case) and they have the expressions

$$\begin{aligned} v_1(\lambda) &= \lambda_1, \\ v_3(\lambda) &= \lambda_5(\lambda_3 - \lambda_6), \\ v_5(\lambda) &= \lambda_2\lambda_4(\lambda_3 - \lambda_6)(\lambda_4 + 5\lambda_3 - 5\lambda_6), \\ v_7(\lambda) &= \lambda_2\lambda_4(\lambda_3 - \lambda_6)^2(\lambda_3\lambda_6 - 2\lambda_6^2 - \lambda_2^2). \end{aligned}$$

For further use we write here again equation (7) for our four Poincaré–Liapunov constants

$$v_{2j+1}(\lambda(\varepsilon)) = \sum_{r \geq 1} \bar{v}_{2j+1,r} \varepsilon^r, \quad j = \overline{0, 3},$$

and

$$\lambda_i(\varepsilon) = \sum_{j \geq 0} \lambda_{i,j} \varepsilon^j.$$

Denote by $hB_1(h)$, $h^3B_3(h)$, $h^5B_5(h)$, $h^7B_7(h)$ the four Bautin functions for the family of quadratic systems in the standard Bautin form. Applying Theorem 4 we have that the first Melnikov function has the expression

$$M_1(h) = \bar{v}_{1,1}hB_1(h) + \bar{v}_{3,1}h^3B_3(h) + \bar{v}_{5,1}h^5B_5(h) + \bar{v}_{7,1}h^7B_7(h).$$

Moreover, if $M_j(h) \equiv 0$ for $j < k$, then

$$M_k(h) = \bar{v}_{1,k}hB_1(h) + \bar{v}_{3,k}h^3B_3(h) + \bar{v}_{5,k}h^5B_5(h) + \bar{v}_{7,k}h^7B_7(h).$$

The reals $\bar{v}_{1,k}$, $\bar{v}_{3,k}$, $\bar{v}_{5,k}$, $\bar{v}_{7,k}$ will be called here the coefficients of the Melnikov function M_k .

We remind that [2], for $\varepsilon = 0$ (i.e. for $\lambda(0) = (\lambda_{1,0}, \dots, \lambda_{6,0})$), system (10) has a center at the origin if and only if one of the following relations holds (first we indicate the name used in literature for the corresponding center condition)

- (a) Lotka–Volterra: $\lambda_{3,0} = \lambda_{6,0}$;
- (b) Symmetric (or Reversible): $\lambda_{2,0} = \lambda_{5,0} = 0$;
- (c) Hamiltonian: $\lambda_{4,0} = \lambda_{5,0} = 0$;
- (d) Darboux (or Codimension 4): $\lambda_{5,0} = \lambda_{4,0} + 5\lambda_{3,0} - 5\lambda_{6,0} = \lambda_{3,0}\lambda_{6,0} - 2\lambda_{6,0}^2 - \lambda_{2,0}^2 = 0$.

In the next lemma we give the expressions of the coefficients of the Melnikov functions, the essential order and the essential parameters for all possible positions of a point $(\lambda_{1,0}, \dots, \lambda_{6,0})$ in the center variety. This lemma is followed by a theorem which gives the essential perturbation and the essential Melnikov function in each situation.

Lemma 5. *For any integer $k \geq 1$, the following statements hold.*

- (i) *Generic Lotka–Volterra: $\lambda_{1,0} = \lambda_{3,0} - \lambda_{6,0} = 0$ and $\lambda_{5,0} \neq 0$.*

If $\bar{v}_{1,j} = \bar{v}_{3,j} = 0$ for $j = \overline{0, k-1}$, then

$$\begin{aligned}\bar{v}_{1,k} &= \lambda_{1,k}, \\ \bar{v}_{3,k} &= \lambda_{5,0}(\lambda_{3,k} - \lambda_{6,k}), \\ \bar{v}_{5,k} &= \lambda_{2,0}\lambda_{4,0}^2(\lambda_{3,k} - \lambda_{6,k}), \\ \bar{v}_{7,k} &= 0.\end{aligned}$$

The essential order is $k^* = 1$ and the essential parameters can be chosen to be $\lambda_{1,1}$ and $\lambda_{6,1}$.

- (ii) *Generic symmetric:* $\lambda_{1,0} = \lambda_{2,0} = \lambda_{5,0} = 0$, $\lambda_{4,0}(\lambda_{3,0} - \lambda_{6,0}) \neq 0$ and $(\lambda_{4,0} + 5\lambda_{3,0} - 5\lambda_{6,0})^2 + (\lambda_{3,0}\lambda_{6,0} - 2\lambda_{6,0}^2)^2 \neq 0$.

If $\bar{v}_{1,j} = \bar{v}_{3,j} = \bar{v}_{5,j} = \bar{v}_{7,j} = 0$ for $j = \overline{0, k-1}$, then

$$\begin{aligned}\bar{v}_{1,k} &= \lambda_{1,k}, \\ \bar{v}_{3,k} &= (\lambda_{3,0} - \lambda_{6,0})\lambda_{5,k}, \\ \bar{v}_{5,k} &= \lambda_{4,0}^2(\lambda_{3,0} - \lambda_{6,0})(\lambda_{4,0} + 5\lambda_{3,0} - 5\lambda_{6,0})\lambda_{2,k}, \\ \bar{v}_{7,k} &= \lambda_{4,0}(\lambda_{3,0} - \lambda_{6,0})^2(\lambda_{3,0}\lambda_{6,0} - 2\lambda_{6,0}^2)\lambda_{2,k}.\end{aligned}$$

The essential order is $k^* = 1$ and the essential parameters can be chosen to be $\lambda_{1,1}$, $\lambda_{2,1}$ and $\lambda_{5,1}$.

- (iii) *Generic Hamiltonian:* $\lambda_{1,0} = \lambda_{4,0} = \lambda_{5,0} = 0$ and $\lambda_{2,0}(\lambda_{3,0} - \lambda_{6,0}) \neq 0$.

If $\bar{v}_{1,j} = \bar{v}_{3,j} = \bar{v}_{5,j} = 0$ for $j = \overline{0, k-1}$, then

$$\begin{aligned}\bar{v}_{1,k} &= \lambda_{1,k}, \\ \bar{v}_{3,k} &= (\lambda_{3,0} - \lambda_{6,0})\lambda_{5,k}, \\ \bar{v}_{5,k} &= \lambda_{2,0}(\lambda_{3,0} - \lambda_{6,0})^2\lambda_{4,k}, \\ \bar{v}_{7,k} &= \lambda_{2,0}(\lambda_{3,0} - \lambda_{6,0})^2(\lambda_{3,0}\lambda_{6,0} - 2\lambda_{6,0}^2 - \lambda_{2,0}^2)\lambda_{4,k}.\end{aligned}$$

The essential order is $k^* = 1$ and the essential parameters can be chosen to be $\lambda_{1,1}$, $\lambda_{4,1}$ and $\lambda_{5,1}$.

- (iv) *Generic Darboux:* $\lambda_{1,0} = \lambda_{5,0} = \lambda_{4,0} + 5\lambda_{3,0} - 5\lambda_{6,0} = \lambda_{3,0}\lambda_{6,0} - 2\lambda_{6,0}^2 - \lambda_{2,0}^2 = 0$ and $\lambda_{2,0}\lambda_{4,0}(\lambda_{3,0} - \lambda_{6,0}) \neq 0$. Then

$$\begin{aligned}\bar{v}_{1,1} &= \lambda_{1,1}, \\ \bar{v}_{3,1} &= (\lambda_{3,0} - \lambda_{6,0})\lambda_{5,1}, \\ \bar{v}_{5,1} &= \lambda_{2,0}\lambda_{4,0}(\lambda_{3,0} - \lambda_{6,0})(\lambda_{4,1} + 5\lambda_{3,1} - 5\lambda_{6,1}), \\ \bar{v}_{7,1} &= \lambda_{2,0}\lambda_{4,0}(\lambda_{3,0} - \lambda_{6,0})^2(\lambda_{3,0}\lambda_{6,1} + \lambda_{6,0}\lambda_{3,1} - 4\lambda_{6,0}\lambda_{6,1} - 2\lambda_{2,0}\lambda_{2,1}).\end{aligned}$$

The essential order is $k^* = 1$ and the essential parameters can be chosen to be $\lambda_{1,1}$, $\lambda_{2,1}$, $\lambda_{4,1}$ and $\lambda_{5,1}$.

- (v) *Symmetric Lotka–Volterra:* $\lambda_{1,0} = \lambda_{2,0} = \lambda_{5,0} = \lambda_{3,0} - \lambda_{6,0} = 0$ and $\lambda_{4,0} \neq 0$.

Then $\bar{v}_{1,1} = \lambda_{1,1}$ and $\bar{v}_{3,1} = \bar{v}_{5,1} = \bar{v}_{7,1} = 0$. If $\lambda_{1,1} = 0$, then

$$\begin{aligned}\bar{v}_{1,2} &= \lambda_{1,2}, \\ \bar{v}_{3,2} &= (\lambda_{3,1} - \lambda_{6,1})\lambda_{5,1}, \\ \bar{v}_{5,2} &= \lambda_{4,0}^2\lambda_{2,1}(\lambda_{3,1} - \lambda_{6,1}), \\ \bar{v}_{7,2} &= 0.\end{aligned}$$

If $\bar{v}_{5,j} = 0$ for $j = \overline{0, k-1}$ with $k \geq 2$, then

$$\bar{v}_{7,k} = 0.$$

The essential order is $k^* = 2$, and the essential parameters can be chosen to be $\lambda_{1,2}$, $\lambda_{2,1}$ and $\lambda_{5,1}$, taking $\lambda_{3,1} = 1$.

(vi) *Symmetric Hamiltonian:* $\lambda_{1,0} = \lambda_{2,0} = \lambda_{4,0} = \lambda_{5,0} = 0$ and $\lambda_{3,0} - \lambda_{6,0} \neq 0$.

Then $\bar{v}_{1,1} = \lambda_{1,1}$, $\bar{v}_{3,1} = (\lambda_{3,0} - \lambda_{6,0})\lambda_{5,1}$, $\bar{v}_{5,1} = \bar{v}_{7,1} = 0$.

If $\bar{v}_{1,1} = \bar{v}_{3,1} = 0$ then

$$\begin{aligned}\bar{v}_{1,2} &= \lambda_{1,2}, \\ \bar{v}_{3,2} &= (\lambda_{3,0} - \lambda_{6,0})\lambda_{5,2}, \\ \bar{v}_{5,2} &= (\lambda_{3,0} - \lambda_{6,0})^2\lambda_{2,1}\lambda_{4,1}, \\ \bar{v}_{7,2} &= (\lambda_{3,0} - \lambda_{6,0})^2(\lambda_{3,0}\lambda_{6,0} - 2\lambda_{6,0}^2)\lambda_{2,1}\lambda_{4,1}.\end{aligned}$$

Moreover, if $\bar{v}_{5,j} = 0$ for $j = \overline{1, k-1}$, with $k \geq 2$ then there exists some $i \in \{1, 2, \dots, k-1\}$ such that

$$\begin{aligned}\bar{v}_{5,k} &= (\lambda_{3,0} - \lambda_{6,0})^2\lambda_{2,i}\lambda_{4,k-i}, \\ \bar{v}_{7,k} &= (\lambda_{3,0} - \lambda_{6,0})^2(\lambda_{3,0}\lambda_{6,0} - 2\lambda_{6,0}^2)\lambda_{2,i}\lambda_{4,k-i}.\end{aligned}$$

The essential order is $k^* = 2$ and the essential parameters can be chosen to be $\lambda_{1,2}$, $\lambda_{4,1}$ and $\lambda_{5,2}$, taking $\lambda_{2,1} = 1$.

(vii) *Symmetric Darboux:* $\lambda_{1,0} = \lambda_{2,0} = \lambda_{5,0} = \lambda_{4,0} + 5\lambda_{3,0} - 5\lambda_{6,0} = \lambda_{3,0}\lambda_{6,0} - 2\lambda_{6,0}^2 = 0$ and $\lambda_{4,0} \neq 0$.

Then $\lambda_{3,0} - \lambda_{6,0} \neq 0$, $\lambda_{3,0} \neq 0$, and

$$\bar{v}_{1,1} = \lambda_{1,1}, \quad \bar{v}_{3,1} = (\lambda_{3,0} - \lambda_{6,0})\lambda_{5,1}, \quad \bar{v}_{5,1} = \bar{v}_{7,1} = 0.$$

If $\bar{v}_{1,1} = \bar{v}_{3,1} = 0$ then

$$\begin{aligned}\bar{v}_{1,2} &= \lambda_{1,2}, \\ \bar{v}_{3,2} &= (\lambda_{3,0} - \lambda_{6,0})\lambda_{5,2}, \\ \bar{v}_{5,2} &= \lambda_{4,0}(\lambda_{3,0} - \lambda_{6,0})\lambda_{2,1}(\lambda_{4,1} + 5\lambda_{3,1} - 5\lambda_{6,1}), \\ \bar{v}_{7,2} &= \lambda_{4,0}(\lambda_{3,0} - \lambda_{6,0})^2\lambda_{2,1}(\lambda_{3,0}\lambda_{6,1} + \lambda_{6,0}\lambda_{3,1}).\end{aligned}$$

The essential order is $k^* = 2$ and the essential parameters can be chosen to be $\lambda_{1,2}$, $\lambda_{5,2}$, $\lambda_{4,1}$ and $\lambda_{6,1}$ taking $\lambda_{2,1} = 1$.

(viii) *Hamiltonian Lotka-Volterra:* $\lambda_{1,0} = \lambda_{4,0} = \lambda_{5,0} = \lambda_{3,0} - \lambda_{6,0} = 0$ and $\lambda_{2,0} \neq 0$. Then $\bar{v}_{1,1} = \lambda_{1,1}$, and $\bar{v}_{3,1} = \bar{v}_{5,1} = \bar{v}_{7,1} = 0$. If $\bar{v}_{1,1} = 0$ then

$$\bar{v}_{1,2} = \lambda_{1,2}, \quad \bar{v}_{3,2} = (\lambda_{3,1} - \lambda_{6,1})\lambda_{5,1}, \quad \bar{v}_{5,2} = \bar{v}_{7,2} = 0.$$

If $\lambda_{3,1} - \lambda_{6,1} \neq 0$ and $\lambda_{1,2} = \lambda_{5,1} = 0$, then $\bar{v}_{1,2} = \bar{v}_{3,2} = 0$ and

$$\begin{aligned}\bar{v}_{1,3} &= \lambda_{1,3}, \\ \bar{v}_{3,3} &= (\lambda_{3,1} - \lambda_{6,1})\lambda_{5,2}, \\ \bar{v}_{5,3} &= \lambda_{2,0}(\lambda_{3,1} - \lambda_{6,1})(\lambda_{4,1} + 5\lambda_{3,1} - 5\lambda_{6,1})\lambda_{4,1}, \\ \bar{v}_{7,3} &= \lambda_{2,0}(\lambda_{6,0}^2 + \lambda_{2,0}^2)(\lambda_{3,1} - \lambda_{6,1})^2\lambda_{4,1}.\end{aligned}$$

The essential order is $k^* = 3$ and the essential parameters can be chosen to be $\lambda_{1,3}$, $\lambda_{3,1}$, $\lambda_{5,2}$ and $\lambda_{4,1}$.

Moreover if $\bar{v}_{1,3} = \bar{v}_{3,3} = \bar{v}_{5,3} = \bar{v}_{7,3} = 0$ then

$$\begin{aligned}\bar{v}_{1,4} &= \lambda_{1,4}, \\ \bar{v}_{3,4} &= (\lambda_{3,1} - \lambda_{6,1})\lambda_{5,3}, \\ \bar{v}_{5,4} &= \lambda_{2,0}(\lambda_{3,1} - \lambda_{6,1})(5\lambda_{3,1} - 5\lambda_{6,1})\lambda_{4,2}, \\ \bar{v}_{7,4} &= \lambda_{2,0}(\lambda_{6,0}^2 + \lambda_{2,0}^2)(\lambda_{3,1} - \lambda_{6,1})^2\lambda_{4,2}.\end{aligned}$$

and if $\lambda_{1,2} = \lambda_{1,3} = \lambda_{1,4} = \lambda_{3,1} - \lambda_{6,1} = \lambda_{5,1} = \lambda_{3,2} - \lambda_{6,2} = 0$ then $\bar{v}_{1,j} = \bar{v}_{3,j} = \bar{v}_{5,j} = \bar{v}_{7,j} = 0$ for $j = \overline{1,4}$ and

$$\begin{aligned}\bar{v}_{1,5} &= \lambda_{1,5}, \\ \bar{v}_{3,5} &= (\lambda_{3,3} - \lambda_{6,3})\lambda_{5,2}, \\ \bar{v}_{5,5} &= \lambda_{2,0}\lambda_{4,1}^2(\lambda_{3,3} - \lambda_{6,3}), \\ \bar{v}_{7,5} &= 0.\end{aligned}$$

(ix) *Symmetric Hamiltonian Lotka–Volterra (Hamiltonian triangle):* $\lambda_{1,0} = \lambda_{2,0} = \lambda_{4,0} = \lambda_{5,0} = \lambda_{3,0} - \lambda_{6,0} = 0$ and $\lambda_{6,0} \neq 0$. Then

$$\begin{aligned}\bar{v}_{3,1} &= \bar{v}_{5,1} = \bar{v}_{7,1} = 0, \\ \bar{v}_{3,2} &= (\lambda_{3,1} - \lambda_{6,1})\lambda_{5,1}, \quad \bar{v}_{5,2} = \bar{v}_{7,2} = 0, \\ \bar{v}_{5,3} &= \bar{v}_{7,3} = 0.\end{aligned}$$

If $\lambda_{3,1} - \lambda_{6,1} \neq 0$ and $\bar{v}_{1,j} = \bar{v}_{3,j} = 0$ for $j = \overline{1,4}$ then

$$\begin{aligned}\bar{v}_{1,4} &= \lambda_{1,4}, \\ \bar{v}_{3,4} &= (\lambda_{3,1} - \lambda_{6,1})\lambda_{5,3}, \\ \bar{v}_{5,4} &= \lambda_{2,1}\lambda_{4,1}(\lambda_{3,1} - \lambda_{6,1})(\lambda_{4,1} + 5\lambda_{3,1} - 5\lambda_{6,1}), \\ \bar{v}_{7,4} &= \lambda_{6,0}^2\lambda_{2,1}\lambda_{4,1}(\lambda_{3,1} - \lambda_{6,1})^2.\end{aligned}$$

The essential order is $k^* = 4$ and the essential parameters can be chosen to be $\lambda_{1,4}$, $\lambda_{5,3}$, $\lambda_{2,1}$ and $\lambda_{4,1}$ taking $\lambda_{3,1} = 1$. If $\lambda_{3,1} - \lambda_{6,1} = \lambda_{5,1} = \lambda_{5,2} = 0$ and $\bar{v}_{1,j} = 0$ for $j = \overline{1,4}$ then $\bar{v}_{3,j} = \bar{v}_{5,j} = \bar{v}_{7,j} = 0$ for $j = \overline{1,4}$ and

$$\begin{aligned}\bar{v}_{1,5} &= \lambda_{1,5}, \\ \bar{v}_{3,5} &= (\lambda_{3,2} - \lambda_{6,2})\lambda_{5,3}, \\ \bar{v}_{5,5} &= \lambda_{2,1}\lambda_{4,1}^2(\lambda_{3,2} - \lambda_{6,2}), \\ \bar{v}_{7,5} &= 0.\end{aligned}$$

(x) *Linear center:* $\lambda_{1,0} = \lambda_{2,0} = \lambda_{3,0} = \lambda_{4,0} = \lambda_{5,0} = \lambda_{6,0} = 0$. Then $\bar{v}_{1,1} = \lambda_{1,1}$ and $\bar{v}_{3,1} = \bar{v}_{5,1} = \bar{v}_{7,1} = 0$. If $\bar{v}_{1,1} = 0$, then $\bar{v}_{1,2} = \lambda_{1,2}$, $\bar{v}_{3,2} = \lambda_{5,1}(\lambda_{3,1} - \lambda_{6,1})$ and $\bar{v}_{5,2} = \bar{v}_{7,2} = 0$.

If $\lambda_{3,1} - \lambda_{6,1} \neq 0$ and $\lambda_{1,2} = \lambda_{5,1} = 0$, then $\bar{v}_{1,2} = \bar{v}_{3,2} = 0$ and $\bar{v}_{1,3} = \lambda_{1,3}$, $\bar{v}_{3,3} = \lambda_{5,2}(\lambda_{3,1} - \lambda_{6,1})$ and $\bar{v}_{5,3} = \bar{v}_{7,3} = 0$.

If $\lambda_{3,1} - \lambda_{6,1} \neq 0$ and $\lambda_{1,3} = \lambda_{5,2} = 0$, then $\bar{v}_{1,3} = \bar{v}_{3,3} = 0$ and $\bar{v}_{1,4} = \lambda_{1,4}$, $\bar{v}_{3,4} = \lambda_{5,3}(\lambda_{3,1} - \lambda_{6,1})$, $\bar{v}_{5,4} = \lambda_{2,1}\lambda_{4,1}(\lambda_{3,1} - \lambda_{6,1})(\lambda_{4,1} + 5(\lambda_{3,1} - \lambda_{6,1}))$ and $\bar{v}_{7,4} = 0$.

If $(\lambda_{3,1} - \lambda_{6,1})\lambda_{2,1}\lambda_{4,1} \neq 0$, $\lambda_{1,4} = \lambda_{5,3} = 0$, $\lambda_{4,1} = 5(\lambda_{6,1} - \lambda_{3,1})$ then $\bar{v}_{1,4} = \bar{v}_{3,4} = \bar{v}_{5,4} = 0$ and $\bar{v}_{1,5} = \lambda_{1,5}$, $\bar{v}_{3,5} = \lambda_{5,4}(\lambda_{3,1} - \lambda_{6,1})$, $\bar{v}_{5,5} = \lambda_{2,1}\lambda_{4,1}(\lambda_{3,1} - \lambda_{6,1})(\lambda_{4,2} + 5(\lambda_{3,2} - \lambda_{6,2}))$ and $\bar{v}_{7,5} = 0$.

If $(\lambda_{3,1} - \lambda_{6,1})\lambda_{2,1}\lambda_{4,1} \neq 0$, $\lambda_{1,5} = \lambda_{5,4} = 0$, $\lambda_{4,2} = 5(\lambda_{6,2} - \lambda_{3,2})$ then $\bar{v}_{1,5} = \bar{v}_{3,5} = \bar{v}_{5,5} = 0$ and

$$\begin{aligned}\bar{v}_{1,6} &= \lambda_{1,6}, \\ \bar{v}_{3,6} &= (\lambda_{3,1} - \lambda_{6,1})\lambda_{5,5}, \\ \bar{v}_{5,6} &= \lambda_{2,1}(\lambda_{3,1} - \lambda_{6,1})^2(\lambda_{4,3} + 5(\lambda_{3,3} - \lambda_{6,3})), \\ \bar{v}_{7,5} &= \lambda_{2,1}(\lambda_{3,1} - \lambda_{6,1})^3(\lambda_{2,1}^2 - \lambda_{3,1}\lambda_{6,1} + 2\lambda_{6,1}^2).\end{aligned}$$

The essential order is $k^* = 6$ and the essential parameters can be chosen to be $\lambda_{1,6}$, $\lambda_{5,5}$, $\lambda_{4,3}$ and $\lambda_{2,1}$ taking $\lambda_{3,1} = \lambda_{6,1} + 1$, $\lambda_{3,3} = \lambda_{6,3}$ and $\lambda_{6,1} = 1/4$.

Proof. The cases (i) and (ii) were already proved in [6].

(iii) Note that $\bar{v}_{3,j} = \bar{v}_{5,j} = 0$ for $j = \overline{0, k-1}$ means that $\lambda_{5,j} = \lambda_{4,j} = 0$ for $j = \overline{0, k-1}$. Then

$$\begin{aligned}\bar{v}_3(\lambda(\varepsilon)) &= [(\lambda_{3,0} - \lambda_{6,0}) + O(\varepsilon)] [\lambda_{5,k}\varepsilon^k + O(\varepsilon^{k+1})], \\ \bar{v}_5(\lambda(\varepsilon)) &= [\lambda_{2,0}(\lambda_{3,0} - \lambda_{6,0})(5\lambda_{3,0} - 5\lambda_{6,0}) + O(\varepsilon)] [\lambda_{4,k}\varepsilon^k + O(\varepsilon^{k+1})], \\ \bar{v}_7(\lambda(\varepsilon)) &= [\lambda_{2,0}(\lambda_{3,0} - \lambda_{6,0})^2(\lambda_{3,0}\lambda_{6,0} - 2\lambda_{6,0}^2 - \lambda_{2,0}^2) + O(\varepsilon)] [\lambda_{4,k}\varepsilon^k + O(\varepsilon^{k+1})]\end{aligned}$$

and the expressions of $\bar{v}_{3,k}$, $\bar{v}_{5,k}$ and $\bar{v}_{7,k}$ easily follow. It is not difficult to see that the range of the map (9) is the same for each k , hence the essential order is $k^* = 1$.

(iv) In this case the range of (9) for $k = 1$ is \mathbb{R}^4 , the largest possible. Hence the essential order is $k^* = 1$.

(v) In this case we have

$$\begin{aligned}\bar{v}_3(\lambda(\varepsilon)) &= [\varepsilon(\lambda_{3,1} - \lambda_{6,1}) + O(\varepsilon^2)] [\varepsilon\lambda_{5,1} + O(\varepsilon^2)], \\ \bar{v}_5(\lambda(\varepsilon)) &= [\varepsilon\lambda_{2,1} + O(\varepsilon^2)] [\varepsilon(\lambda_{3,1} - \lambda_{6,1}) + O(\varepsilon^2)] [\lambda_{4,0}^2 + O(\varepsilon)], \\ \bar{v}_7(\lambda(\varepsilon)) &= [\varepsilon\lambda_{2,1} + O(\varepsilon^2)] [\varepsilon(\lambda_{3,1} - \lambda_{6,1}) + O(\varepsilon^2)]^2 [-\lambda_{4,0}\lambda_{3,0}^2 + O(\varepsilon)].\end{aligned}$$

The coefficient of ε in each of the above expressions is null, and it is easy to identify the coefficient of ε^2 .

In order to see that if $\bar{v}_{5,j} = 0$ for $j = 0, \dots, k-1$ then $\bar{v}_{7,k} = 0$, we note that $\bar{v}_7(\lambda(\varepsilon)) = \bar{v}_5(\lambda(\varepsilon)) [\varepsilon(\lambda_{3,1} - \lambda_{6,1}) + O(\varepsilon^2)] (-\frac{\lambda_{3,0}^2}{\lambda_{4,0}} + O(\varepsilon))$. We have that if $\bar{v}_{5,j} = 0$ for $j = 0, \dots, k-1$ then $\bar{v}_5(\lambda(\varepsilon))$ has order k in $\varepsilon = 0$. Hence $\bar{v}_7(\lambda(\varepsilon))$ has at least order $k+1$ in $\varepsilon = 0$, meaning that $\bar{v}_{7,k} = 0$.

Since the image of (9) for $k = 2$ is $\mathcal{R} = \{(a, b, c, 0) : (a, b, c) \in \mathbb{R}^3\}$, from what we showed above we deduce that for $k \neq 2$ the image of (9) is either equal or

contained in \mathcal{R} . Taking all these into account we deduce that the essential order is $k^* = 2$.

(vi) In this case we have

$$\begin{aligned}\bar{v}_3(\lambda(\varepsilon)) &= [(\lambda_{3,0} - \lambda_{6,0}) + O(\varepsilon)] [\varepsilon\lambda_{5,1} + \varepsilon^2\lambda_{5,2} + O(\varepsilon^2)], \\ \bar{v}_5(\lambda(\varepsilon)) &= [\varepsilon\lambda_{2,1} + O(\varepsilon^2)] [\varepsilon\lambda_{4,1} + O(\varepsilon^2)] [(\lambda_{3,0} - \lambda_{6,0})^2 + O(\varepsilon)], \\ \bar{v}_7(\lambda(\varepsilon)) &= [\varepsilon\lambda_{2,1} + O(\varepsilon^2)] [\varepsilon\lambda_{4,1} + O(\varepsilon^2)] \\ &\quad [(\lambda_{3,0} - \lambda_{6,0})^2(\lambda_{3,0}\lambda_{6,0} - 2\lambda_{6,0}^2) + O(\varepsilon)].\end{aligned}$$

Identifying the coefficients of ε and ε^2 we obtain in the expressions of $\bar{v}_{j,1}$ and $\bar{v}_{j,2}$ for $j = 1, 3, 5, 7$ given in the statement.

Assume that $\bar{v}_{5,j} = 0$ for $j = \overline{1, k-1}$. Then, from the above expressions we deduce that $\lambda_2(\varepsilon)\lambda_4(\varepsilon)$ has order k in $\varepsilon = 0$. Hence, if i is such that $\lambda_2(\varepsilon) = \varepsilon^i\lambda_{2,i} + O(\varepsilon^{i+1})$ then $\lambda_4(\varepsilon) = \varepsilon^{k-i}\lambda_{4,k-i} + O(\varepsilon^{k-i+1})$ and the coefficient of ε^k in their product is $\lambda_{2,i}\lambda_{4,k-i}$. The expressions of $\bar{v}_{5,k}$ and $\bar{v}_{7,k}$ follow from the above considerations.

(vii) Taking $\lambda_{2,1} = 1$ and $\lambda_{3,1} = 0$ the range of the map (9) for $k = 2$ is \mathbb{R}^4 , the largest possible. Hence indeed $k^* = 2$ is the essential order and $\lambda_{1,2}, \lambda_{4,1}, \lambda_{5,2}, \lambda_{6,1}$ are the essential parameters.

(viii) In this case we have

$$\begin{aligned}\bar{v}_3(\lambda(\varepsilon)) &= [(\lambda_{3,1} - \lambda_{6,1})\varepsilon + O(\varepsilon^2)] [\lambda_{5,1}\varepsilon + \lambda_{5,2}\varepsilon^2 + \lambda_{5,3}\varepsilon^3 + O(\varepsilon^4)], \\ \bar{v}_5(\lambda(\varepsilon)) &= [\lambda_{2,0}(\lambda_{3,1} - \lambda_{6,1})(\lambda_{4,1} + 5\lambda_{3,1} - 5\lambda_{6,1})\varepsilon^2 + O(\varepsilon^3)] \\ &\quad [\lambda_{4,1}\varepsilon + \lambda_{4,2}\varepsilon^2 + O(\varepsilon^3)], \\ \bar{v}_7(\lambda(\varepsilon)) &= [\lambda_{2,0}(\lambda_{3,1} - \lambda_{6,1})^2(-\lambda_{6,0}^2 - \lambda_{2,0}^2)\varepsilon^2 + O(\varepsilon^3)] \\ &\quad [\lambda_{4,1}\varepsilon + \lambda_{4,2}\varepsilon^2 + O(\varepsilon^3)].\end{aligned}$$

Identifying the coefficients of ε , ε^2 , ε^3 , ε^4 and ε^5 we obtain the expressions of $\bar{v}_{i,j}$ given in the statement. If $\lambda_{3,1} - \lambda_{6,1} \neq 0$ and $\lambda_{1,2} = \lambda_{5,1} = 0$ then the closure of the range of the map (9) for $k^* = 3$ is \mathbb{R}^4 . In fact the range is $\mathbb{R}^4 \setminus \mathcal{R}_1 \cup \mathcal{R}_2$ where $\mathcal{R}_1 = \{(a, b, c, d) \in \mathbb{R}^4 \mid d \neq 0, (\lambda_{6,0}^2 + \lambda_{2,0}^2)c - 5d = 0\}$ and $\mathcal{R}_2 = \{(a, b, c, d) \in \mathbb{R}^4 \mid c \neq 0, d = 0\}$. We continue our study giving other cases that recover the gaps of the previous range. If $\lambda_{3,1} - \lambda_{6,1} \neq 0$ and $\bar{v}_{1,j} = \bar{v}_{3,j} = \bar{v}_{5,j} = \bar{v}_{7,j} = 0$ for $j = \overline{1, 3}$ then the range of the map (9) for $k^* = 4$ is \mathcal{R}_1 . If $\lambda_{3,1} - \lambda_{6,1} = \lambda_{5,1} = \lambda_{3,2} - \lambda_{6,2} = 0$ and $\bar{v}_{1,j} = \bar{v}_{3,j} = \bar{v}_{5,j} = \bar{v}_{7,j} = 0$ for $j = \overline{1, 4}$ then the range of the map (9) for $k^* = 5$ is \mathcal{R}_2 .

(ix) By the same reasonings of the preceding cases we obtain the expressions of $\bar{v}_{i,j}$ given in the statement. If $\lambda_{3,1} - \lambda_{6,1} \neq 0$ and $\bar{v}_{1,j} = \bar{v}_{3,j} = 0$ for $j = \overline{1, 4}$ then the closure of the range of the map (9) for $k^* = 3$ is \mathbb{R}^4 . In fact the range is $\mathbb{R}^4 \setminus \mathcal{R}_1$ where $\mathcal{R}_1 = \{(a, b, c, d) \in \mathbb{R}^4 \mid c \neq 0, d = 0\}$. We continue our study giving the case that recover the gap of the previous range. If $\lambda_{3,1} - \lambda_{6,1} = \lambda_{5,1} = \lambda_{5,2} = 0$ and

$\bar{v}_{1,j} = 0$ for $j = \overline{1,4}$ then $\bar{v}_{3,j} = \bar{v}_{5,j} = \bar{v}_{7,j} = 0$ for $j = \overline{1,4}$ and the range of the map (9) for $k^* = 5$ is \mathcal{R}_1 .

(x) In this case we have

$$\begin{aligned}\bar{v}_3(\lambda(\varepsilon)) &= [(\lambda_{3,1} - \lambda_{6,1})\varepsilon + O(\varepsilon^2)] \\ &\quad [\lambda_{5,1}\varepsilon + \lambda_{5,2}\varepsilon^2 + \lambda_{5,3}\varepsilon^3 + \lambda_{5,4}\varepsilon^4 + \lambda_{5,5}\varepsilon^5 + O(\varepsilon^6)], \\ \bar{v}_5(\lambda(\varepsilon)) &= [\lambda_{2,1}\varepsilon + O(\varepsilon^2)] [\lambda_{4,1}\varepsilon + O(\varepsilon^2)] [(\lambda_{3,1} - \lambda_{6,1})\varepsilon + O(\varepsilon^2)] \\ &\quad [(\lambda_{4,1} + 5(\lambda_{3,1} - \lambda_{6,1}))\varepsilon + (\lambda_{4,2} + 5(\lambda_{3,2} - \lambda_{6,2}))\varepsilon^2 \\ &\quad + (\lambda_{4,3} + 5(\lambda_{3,3} - \lambda_{6,3}))\varepsilon^3 + O(\varepsilon^4)], \\ \bar{v}_7(\lambda(\varepsilon)) &= [\lambda_{2,1}\varepsilon + O(\varepsilon^2)] [\lambda_{4,1}\varepsilon + O(\varepsilon^2)] [(\lambda_{3,1} - \lambda_{6,1})\varepsilon + O(\varepsilon^2)] \\ &\quad [(\lambda_{3,1}\lambda_{6,1} - 2\lambda_{6,1}^2 - \lambda_{2,1}^2)\varepsilon^2 + O(\varepsilon^3)].\end{aligned}$$

Identifying the coefficients of ε , ε^2 , ε^3 , ε^4 , ε^5 and ε^6 we obtain the expressions of $\bar{v}_{i,j}$ given in the statement. If $(\lambda_{3,1} - \lambda_{6,1})\lambda_{2,1}\lambda_{4,1} \neq 0$ and $\lambda_{1,j} = 0$ for $j = \overline{1,5}$, $\lambda_{5,j} = 0$ for $j = \overline{1,4}$, $\lambda_{4,j} = 5(\lambda_{6,j} - \lambda_{3,j})$ for $j = 1, 2$, the range of the map (9) for $k^* = 6$ is \mathbb{R}^4 , the largest possible. Hence indeed $k^* = 6$ is the essential order and $\lambda_{1,6}$, $\lambda_{5,5}$, $\lambda_{4,3}$ and $\lambda_{2,1}$ are the essential parameters. \square

Theorem 6. *The essential perturbations and the essential Melnikov function are:*

(i) *Generic Lotka–Volterra center: $\lambda_{1,0} = \lambda_{3,0} - \lambda_{6,0} = 0$ and $\lambda_{5,0} \neq 0$*

$$\begin{aligned}\dot{x} &= -y - \lambda_{6,0}x^2 + (2\lambda_{2,0} + \lambda_{5,0})xy + \lambda_{6,0}y^2 + \varepsilon(\lambda_{1,1}x + \lambda_{6,1}y^2), \\ \dot{y} &= x + \lambda_{2,0}x^2 + (2\lambda_{6,0} + \lambda_{4,0})xy - \lambda_{2,0}y^2 + \varepsilon\lambda_{1,1}y.\end{aligned}$$

The corresponding essential Melnikov function is the first one and it has the form

$$M_1(h) = \lambda_{1,1}hB_1(h) + \lambda_{6,1}h^3\tilde{B}_3(h).$$

(ii) *Generic symmetric center: $\lambda_{1,0} = \lambda_{2,0} = \lambda_{5,0} = 0$, $\lambda_{4,0}(\lambda_{3,0} - \lambda_{6,0}) \neq 0$ and $(\lambda_{4,0} + 5\lambda_{3,0} - 5\lambda_{6,0})^2 + (\lambda_{3,0}\lambda_{6,0} - 2\lambda_{6,0}^2)^2 \neq 0$.*

$$\begin{aligned}\dot{x} &= -y - \lambda_{3,0}x^2 + \lambda_{6,0}y^2 + \varepsilon(\lambda_{1,1}x + (2\lambda_{2,1} + \lambda_{5,1})xy), \\ \dot{y} &= x + (2\lambda_{3,0} + \lambda_{4,0})xy + \varepsilon(\lambda_{1,1}y + \lambda_{2,1}x^2 - \lambda_{2,1}y^2).\end{aligned}$$

The corresponding essential Melnikov function is the first one and it has the form, when $\lambda_{4,0} + 5\lambda_{3,0} - 5\lambda_{6,0} \neq 0$,

$$M_1(h) = \lambda_{1,1}hB_1(h) + \lambda_{5,1}h^3B_3(h) + \lambda_{2,1}h^5\tilde{B}_5(h),$$

and, when $\lambda_{4,0} + 5\lambda_{3,0} - 5\lambda_{6,0} = 0$,

$$M_1(h) = \lambda_{1,1}hB_1(h) + \lambda_{5,1}h^3B_3(h) + \lambda_{2,1}h^7B_7(h).$$

(iii) *Generic Hamiltonian center: $\lambda_{1,0} = \lambda_{4,0} = \lambda_{5,0} = 0$ and $\lambda_{2,0}(\lambda_{3,0} - \lambda_{6,0}) \neq 0$*

$$\begin{aligned}\dot{x} &= -y - \lambda_{3,0}x^2 + 2\lambda_{2,0}xy + \lambda_{6,0}y^2 + \varepsilon(\lambda_{1,1}x + \lambda_{5,1}xy), \\ \dot{y} &= x + \lambda_{2,0}x^2 + 2\lambda_{3,0}xy - \lambda_{2,0}y^2 + \varepsilon(\lambda_{1,1}y + \lambda_{4,1}xy).\end{aligned}$$

The corresponding essential Melnikov function is the first one and it has the form

$$M_1(h) = \lambda_{1,1}hB_1(h) + \lambda_{5,1}h^3B_3(h) + \lambda_{4,1}h^5\tilde{B}_5(h).$$

- (iv) *Generic Darboux center:* $\lambda_{1,0} = \lambda_{5,0} = \lambda_{4,0} + 5\lambda_{3,0} - 5\lambda_{6,0} = \lambda_{3,0}\lambda_{6,0} - 2\lambda_{6,0}^2 - \lambda_{2,0}^2 = 0$ and $\lambda_{2,0}\lambda_{4,0}(\lambda_{3,0} - \lambda_{6,0}) \neq 0$

$$\begin{aligned}\dot{x} &= -y - \lambda_{3,0}x^2 + 2\lambda_{2,0}xy + \lambda_{6,0}y^2 + \varepsilon(\lambda_{1,1}x + (2\lambda_{2,1} + \lambda_{5,1})xy), \\ \dot{y} &= x + \lambda_{2,0}x^2 + (7\lambda_{6,0} - 5\lambda_{3,0})xy - \lambda_{2,0}y^2 + \varepsilon(\lambda_{1,1}y + \lambda_{4,1}xy).\end{aligned}$$

where $(\lambda_{3,0}\lambda_{6,0} - 2\lambda_{6,0}^2 - \lambda_{2,0}^2) = 0$. The corresponding essential Melnikov function is the first one and it has the form

$$M_1(h) = \lambda_{1,1}hB_1(h) + \lambda_{5,1}h^3B_3(h) + \lambda_{4,1}h^5B_5(h) + \lambda_{2,1}h^7B_7(h).$$

- (v) *Symmetric Lotka–Volterra center:* $\lambda_{1,0} = \lambda_{2,0} = \lambda_{5,0} = \lambda_{3,0} - \lambda_{6,0} = 0$ and $\lambda_{4,0} \neq 0$

$$\begin{aligned}\dot{x} &= -y - \lambda_{6,0}x^2 + \lambda_{6,0}y^2 + \varepsilon(2\lambda_{2,1} + \lambda_{5,1})xy + \varepsilon^2\lambda_{1,2}x, \\ \dot{y} &= x + (2\lambda_{6,0} + \lambda_{4,0})xy + 2\varepsilon xy + \varepsilon\lambda_{2,1}(x^2 - y^2) + \varepsilon^2\lambda_{1,2}y.\end{aligned}$$

The corresponding essential Melnikov function is the second one and it has the form

$$M_2(h) = \lambda_{1,2}hB_1(h) + \lambda_{5,1}h^3B_3(h) + \lambda_{2,1}h^5B_5(h).$$

- (vi) *Symmetric Hamiltonian center:* $\lambda_{1,0} = \lambda_{2,0} = \lambda_{4,0} = \lambda_{5,0} = 0$ and $\lambda_{3,0} - \lambda_{6,0} \neq 0$

$$\begin{aligned}\dot{x} &= -y - \lambda_{3,0}x^2 + \lambda_{6,0}y^2 + \varepsilon 2xy + \varepsilon^2(\lambda_{1,2}x + \lambda_{5,2}xy), \\ \dot{y} &= x + 2\lambda_{3,0}xy + \varepsilon(x^2 + \lambda_{4,1}xy - y^2) + \varepsilon^2\lambda_{1,2}y.\end{aligned}$$

The corresponding essential Melnikov function is the second one and it has the form

$$M_2(h) = \lambda_{1,2}hB_1(h) + \lambda_{5,2}h^3B_3(h) + \lambda_{4,1}h^5\tilde{B}_5(h).$$

- (vii) *Symmetric Darboux center:* $\lambda_{1,0} = \lambda_{2,0} = \lambda_{5,0} = \lambda_{4,0} + 5\lambda_{3,0} - 5\lambda_{6,0} = \lambda_{3,0}\lambda_{6,0} - 2\lambda_{6,0}^2 = 0$ and $\lambda_{4,0} \neq 0$

$$\begin{aligned}\dot{x} &= -y - \lambda_{3,0}x^2 + \lambda_{6,0}y^2 + \varepsilon(2xy + \lambda_{6,1}y^2) + \varepsilon^2(\lambda_{1,2}x + \lambda_{5,2}xy), \\ \dot{y} &= x + (5\lambda_{6,0} - 3\lambda_{3,0})xy + \varepsilon(x^2 + \lambda_{4,1}xy - y^2) + \varepsilon^2\lambda_{1,2}y.\end{aligned}$$

The corresponding essential Melnikov function is the second one and it has the form

$$M_2(h) = \lambda_{1,2}hB_1(h) + \lambda_{5,2}h^3B_3(h) + \lambda_{4,1}h^5B_5(h) + \lambda_{6,1}h^7B_7(h).$$

- (viii) *Lotka–Volterra Hamiltonian center:* $\lambda_{1,0} = \lambda_{4,0} = \lambda_{5,0} = \lambda_{3,0} - \lambda_{6,0} = 0$ and $\lambda_{2,0} \neq 0$

$$\begin{aligned}\dot{x} &= -y - \lambda_{6,0}x^2 + 2\lambda_{2,0}xy + \lambda_{6,0}y^2 - \varepsilon\lambda_{3,1}x^2 + \varepsilon^2\lambda_{5,2}xy + \varepsilon^3\lambda_{1,3}x, \\ \dot{y} &= x + \lambda_{2,0}x^2 + 2\lambda_{6,0}xy - \lambda_{2,0}y^2 + \varepsilon(2\lambda_{3,1} + \lambda_{4,1})xy + \varepsilon^3\lambda_{1,3}y.\end{aligned}$$

The corresponding essential Melnikov function is the third one and it has the form

$$M_3(h) = \lambda_{1,3}hB_1(h) + \lambda_{3,1}\lambda_{5,2}h^3B_3(h) + \lambda_{3,1}(\lambda_{4,1} + 5\lambda_{3,1})\lambda_{4,1}h^5B_5(h) + \lambda_{3,1}^2\lambda_{4,1}h^7B_7(h).$$

(ix) *Symmetric Lotka–Volterra Hamiltonian center (or Hamiltonian triangle):*

$$\lambda_{1,0} = \lambda_{2,0} = \lambda_{4,0} = \lambda_{5,0} = \lambda_{3,0} - \lambda_{6,0} = 0 \text{ and } \lambda_{6,0} \neq 0$$

$$\begin{aligned} \dot{x} &= -y - \lambda_{6,0}x^2 + \lambda_{6,0}y^2 - \varepsilon(x^2 + 2\lambda_{2,1}xy) + \varepsilon^3\lambda_{5,3}xy + \varepsilon^4\lambda_{1,4}x, \\ \dot{y} &= x + 2\lambda_{6,0}xy + \varepsilon(\lambda_{2,1}x^2 + (-2 + \lambda_{4,1})xy - \lambda_{2,1}y^2) + \varepsilon^4\lambda_{1,4}y. \end{aligned}$$

The corresponding essential Melnikov function is the fourth one and it has the form

$$M_4(h) = \lambda_{1,4}hB_1(h) + \lambda_{5,3}h^3B_3(h) + \lambda_{2,1}(\lambda_{4,1} + 5)\lambda_{4,1}h^5B_5(h) + \lambda_{2,1}\lambda_{4,1}h^7B_7(h).$$

(x) *Linear center:* $\lambda_{1,0} = \lambda_{2,0} = \lambda_{3,0} = \lambda_{4,0} = \lambda_{5,0} = \lambda_{6,0} = 0$

$$\begin{aligned} \dot{x} &= -y + \varepsilon(-5x^2 + y^2 + 8\lambda_{2,1}xy)/4 + \varepsilon^5\lambda_{5,5}xy + \varepsilon^6\lambda_{1,6}x, \\ \dot{y} &= x + \varepsilon(-5xy + 2\lambda_{2,1}(x^2 - y^2)) + \varepsilon^3\lambda_{4,3}xy + \varepsilon^6\lambda_{1,6}y. \end{aligned}$$

The corresponding essential Melnikov function is the sixth one and it has the form

$$M_6(h) = \lambda_{1,6}hB_1(h) + \lambda_{5,5}h^3B_3(h) + \lambda_{2,1}\lambda_{4,3}h^5B_5(h) + \lambda_{2,1}(16\lambda_{2,1}^2 - 3)h^7B_7(h).$$

We consider that the cases (viii) and (ix) in the above theorem require a discussion. Note that, in the case (viii) the coefficients of the Bautin functions which form $M_3(h)$ vary in some set which is dense in \mathbb{R}^4 , but it is not the whole \mathbb{R}^4 (for details one might see the proof of Lemma 5 (viii)). Anyway, for well chosen values of the parameters there are Melnikov functions whose coefficients vary in the complement of the range of the coefficients of $M_3(h)$. Hence, to study the cyclicity of the period annulus, one can identify the Bautin functions from the expression of $M_3(h)$ and study the zeros of any linear combination of these functions. In the case that the maximum number of zeros (counted with multiplicity) is realized by simple zeros, the cyclicity is found. Otherwise, one can find an upper bound of the cyclicity, but the determination of its exact value, as it is known, is a complicated problem. The same discussion is valid for the case (ix).

3.2. Essential perturbations of linear centers with cubic nonlinearities.

As it was proved by K.S. Sibirsky [28], see also [27] and the references therein, by an affine change of coordinates, any cubic homogeneous center can be written

$$\begin{aligned} \dot{x} &= -y + \lambda x - (\omega + \theta - a)x^3 - (\eta - 3\mu)x^2y \\ &\quad - (3\omega - 3\theta + 2a - \xi)xy^2 - (\mu - \nu)y^3, \\ \dot{y} &= x + \lambda y + (\mu + \nu)x^3 + (3\omega + 3\theta + 2a)x^2y \\ &\quad + (\eta - 3\mu)xy^2 + (\omega - \theta - a)y^3, \end{aligned} \tag{11}$$

where $\lambda, \omega, \theta, a, \eta, \mu, \xi, \nu$ are real parameters.

It can be shown that this family has the following set of Poincaré–Liapunov constants:

$$v_1 = \lambda, \quad v_3 = \xi, \quad v_5 = \nu a, \quad v_7 = \omega\theta a, \quad v_9 = \theta a^2\eta,$$

$$v_{11} = \theta [4(\mu^2 + \theta^2) - a^2] a^2.$$

The center cases of system (11) are the following:

- (I) Hamiltonian: $\lambda = \xi = a = 0$;
- (II) Symmetric: $\lambda = \xi = \nu = \theta = 0$;
- (III) Darboux: $\lambda = \xi = \nu = \omega = \eta = [4(\mu^2 + \theta^2) - a^2] = 0$.

Analogously to the previous subsection, next lemma provides the expressions of the coefficients of the Melnikov functions, the essential order and the essential parameters for all possible positions of a point in the center variety. This lemma is followed by a theorem which gives the essential perturbation and the essential Melnikov function in each situation.

Lemma 7. *For any integer $k > 0$, the following statements hold.*

- (i) *Generic Hamiltonian center: $\lambda_0 = a_0 = \xi_0 = 0$ and $\nu_0^2 + \theta_0^2 \neq 0$.*

- *Case 1: $\nu_0^2 + \omega_0^2 \neq 0$.*

If $\bar{v}_{1,j} = \bar{v}_{3,j} = \bar{v}_{5,j} = \bar{v}_{7,j} = 0$ for $j = \overline{0, k-1}$ then

$$\begin{aligned} \bar{v}_{1,k} &= \lambda_k, \\ \bar{v}_{3,k} &= \xi_k, \\ \bar{v}_{5,k} &= \nu_0 a_k, \\ \bar{v}_{7,k} &= \omega_0 \theta_0 a_k, \\ \bar{v}_{9,k} &= \bar{v}_{11,k} = 0. \end{aligned}$$

The essential order is $k^ = 1$ and the essential parameters can be chosen to be λ_1, ξ_1, a_1 .*

- *Case 2: $\nu_0 = \omega_0 = 0$.*

Then

$$\begin{aligned} \bar{v}_{1,1} &= \lambda_1, \\ \bar{v}_{3,1} &= \xi_1, \\ \bar{v}_{5,1} &= \bar{v}_{7,1} = \bar{v}_{9,1} = \bar{v}_{11,1} = 0. \end{aligned}$$

If $\bar{v}_{1,1} = \bar{v}_{3,1} = 0$ then

$$\begin{aligned} \bar{v}_{1,2} &= \lambda_2, \\ \bar{v}_{3,2} &= \xi_2, \\ \bar{v}_{5,2} &= a_1 \nu_1, \\ \bar{v}_{7,2} &= \theta_0 a_1 \omega_1, \\ \bar{v}_{9,2} &= \eta_0 \theta_0 a_1^2, \\ \bar{v}_{11,2} &= 4\theta_0(\mu_0^2 + \theta_0^2) a_1^2. \end{aligned}$$

If $\bar{v}_{1,j} = \bar{v}_{3,j} = \bar{v}_{5,j} = \bar{v}_{7,j} = \bar{v}_{9,j} = \bar{v}_{11,j} = 0$ for $j = \overline{1, k-1}$ with $k \geq 3$, and

- *k is even, then*

$$\begin{aligned} \bar{v}_{1,k} &= \lambda_k, \\ \bar{v}_{3,k} &= \xi_k, \\ \bar{v}_{5,k} &= a_i \nu_{k-i}, \\ \bar{v}_{7,k} &= \theta_0 a_i \omega_{k-i}, \\ \bar{v}_{9,k} &= \eta_0 \theta_0 a_{k-1}^2, \\ \bar{v}_{11,k} &= 4\theta_0(\mu_0^2 + \theta_0^2) a_{k-1}^2. \end{aligned}$$

- k is odd, then

$$\begin{aligned}
\bar{v}_{1,k} &= \lambda_k, \\
\bar{v}_{3,k} &= \xi_k, \\
\bar{v}_{5,k} &= a_i \nu_{k-i}, \\
\bar{v}_{7,k} &= \theta_0 a_i \omega_{k-i}, \\
\bar{v}_{9,k} &= \bar{v}_{11,k} = 0.
\end{aligned}$$

The essential order is $k^* = 2$ and the essential parameters can be chosen to be $\lambda_2, \xi_2, a_1, \nu_1, \omega_1$.

- (ii) *Generic Symmetric center:* $\lambda_0 = \xi_0 = \nu_0 = \theta_0 = 0$, $a_0 \neq 0$ and $\omega_0^2 + \eta_0^2 + (4\mu_0^2 - a_0^2)^2 \neq 0$.

If $\bar{v}_{1,j} = \bar{v}_{3,j} = \bar{v}_{5,j} = \bar{v}_{7,j} = \bar{v}_{9,j} = \bar{v}_{11,j} = 0$ for $j = \overline{0, k-1}$, then

$$\begin{aligned}
\bar{v}_{1,k} &= \lambda_k, \\
\bar{v}_{3,k} &= \xi_k, \\
\bar{v}_{5,k} &= a_0 \nu_k, \\
\bar{v}_{7,k} &= a_0 \omega_0 \theta_k, \\
\bar{v}_{9,k} &= a_0^2 \eta_0 \theta_k, \\
\bar{v}_{11,k} &= a_0^2 (4\mu_0^2 - a_0^2) \theta_k.
\end{aligned}$$

The essential order is $k^* = 1$ and the essential parameters can be chosen to be $\lambda_1, \xi_1, \nu_1, \theta_1$.

- (iii) *Generic Darboux center:* $\lambda_0 = \xi_0 = \nu_0 = \omega_0 = \eta_0 = 4(\mu_0^2 + \theta_0^2) - a_0^2 = 0$, $a_0 \neq 0$ and $\theta_0 \neq 0$.

Then

$$\begin{aligned}
\bar{v}_{1,1} &= \lambda_1, \\
\bar{v}_{3,1} &= \xi_1, \\
\bar{v}_{5,1} &= a_0 \nu_1, \\
\bar{v}_{7,1} &= a_0 \theta_0 \omega_1, \\
\bar{v}_{9,1} &= a_0^2 \theta_0 \eta_1, \\
\bar{v}_{11,1} &= 2a_0^2 \theta_0 (4\mu_0 \mu_1 + 4\theta_0 \theta_1 - a_0 a_1).
\end{aligned}$$

The essential order is $k^* = 1$ and the essential parameters can be chosen to be $\lambda_1, \xi_1, \nu_1, \omega_1, \eta_1, a_1$.

- (iv) *Hamiltonian symmetric center:* $\lambda_0 = \xi_0 = a_0 = \nu_0 = \theta_0 = 0$ and $\omega_0^2 + \eta_0^2 + \mu_0^2 \neq 0$.

Then

$$\begin{aligned}
\bar{v}_{1,1} &= \lambda_1, \\
\bar{v}_{3,1} &= \xi_1, \\
\bar{v}_{5,1} &= \bar{v}_{7,1} = \bar{v}_{9,1} = \bar{v}_{11,1} = 0.
\end{aligned}$$

If $\bar{v}_{1,1} = \bar{v}_{3,1} = 0$

$$\begin{aligned}\bar{v}_{1,2} &= \lambda_2, \\ \bar{v}_{3,2} &= \xi_2, \\ \bar{v}_{5,2} &= a_1 \nu_1, \\ \bar{v}_{7,2} &= \omega_0 a_1 \theta_1, \\ \bar{v}_{9,2} &= \bar{v}_{11,2} = 0.\end{aligned}$$

If $\bar{v}_{1,j} = \bar{v}_{3,j} = \bar{v}_{5,j} = \bar{v}_{7,j} = \bar{v}_{9,j} = \bar{v}_{11,j} = 0$ for $j = \overline{1, k-1}$ with $k \geq 3$, then $\bar{v}_{9,k} = \bar{v}_{11,k} = 0$.

The essential order is $k^* = 2$ and the essential parameters can be chosen to be $\lambda_2, \xi_2, \nu_1, \theta_1$ taking $a_1 = 1$.

- (v) *Symmetric Darboux center:* $\lambda_0 = \xi_0 = \nu_0 = \theta_0 = \omega_0 = \eta_0 = 4\mu_0^2 - a_0^2 = 0$ and $a_0 \neq 0$. Then

$$\begin{aligned}\bar{v}_{1,1} &= \lambda_1, \\ \bar{v}_{3,1} &= \xi_1, \\ \bar{v}_{5,1} &= a_0 \nu_1, \\ \bar{v}_{7,1} &= \bar{v}_{9,1} = \bar{v}_{11,1} = 0.\end{aligned}$$

If $\bar{v}_{1,1} = \bar{v}_{3,1} = \bar{v}_{5,1} = 0$ then

$$\begin{aligned}\bar{v}_{1,2} &= \lambda_2, \\ \bar{v}_{3,2} &= \xi_2, \\ \bar{v}_{5,2} &= a_0 \nu_2, \\ \bar{v}_{7,2} &= a_0 \theta_1 \omega_1, \\ \bar{v}_{9,2} &= a_0^2 \theta_1 \eta_1, \\ \bar{v}_{11,2} &= a_0^2 \theta_1 (8\mu_0 \mu_1 - 2a_0 a_1).\end{aligned}$$

The essential order is $k^* = 2$ and the essential parameters can be chosen to be $\lambda_2, \xi_2, \nu_2, \omega_1, \eta_1, a_1$ taking $\theta_1 = 1$.

- (vi) *Linear center:* $\lambda_0 = \xi_0 = a_0 = \nu_0 = \theta_0 = \omega_0 = \eta_0 = \mu_0 = 0$.

If $\bar{v}_{1,j} = \bar{v}_{3,j} = \bar{v}_{5,j} = \bar{v}_{7,j} = \bar{v}_{9,j} = \bar{v}_{11,j} = 0$ for $j = 0, 1, 2, 3, 4$, then

$$\begin{aligned}\bar{v}_{1,5} &= \lambda_5, \\ \bar{v}_{3,5} &= \xi_5, \\ \bar{v}_{5,5} &= a_1 \nu_4, \\ \bar{v}_{7,5} &= a_1 \theta_1 \omega_3, \\ \bar{v}_{9,5} &= a_1^2 \theta_1 \eta_2, \\ \bar{v}_{11,5} &= a_1^2 \theta_1 (4(\mu_1^2 + \theta_1^2) - a_1^2).\end{aligned}$$

The essential is $k^* = 5$ and the essential parameters can be chosen to be $\lambda_5, \xi_5, \nu_4, \omega_3, \eta_2, \theta_1$ taking $a_1 = 1$.

Theorem 8. *The essential perturbations and the essential Melnikov functions are:*

- (i) *Generic Hamiltonian center:* $\lambda_0 = a_0 = \xi_0 = 0$ and $\nu_0^2 + \theta_0^2 \neq 0$

– *Case 1:* $\nu_0^2 + \omega_0^2 \neq 0$

$$\begin{aligned}\dot{x} &= -y - (\omega_0 + \theta_0)x^3 - (\eta_0 - 3\mu_0)x^2y - (3\omega_0 - 3\theta_0)xy^2 \\ &\quad - (\mu_0 - \nu_0)y^3 + \varepsilon\lambda_1x + \varepsilon a_1x^3 - \varepsilon(2a_1 - \xi_1)xy^2, \\ \dot{y} &= x + (\mu_0 + \nu_0)x^3 + (3\omega_0 + 3\theta_0)x^2y + (\eta_0 - 3\mu_0)xy^2 \\ &\quad + (\omega_0 - \theta_0)y^3 + \varepsilon\lambda_1y + \varepsilon 2a_1x^2y - \varepsilon a_1y^3.\end{aligned}$$

The corresponding essential Melnikov function is the first one and it has the form

$$M_1(h) = \lambda_1 h B_1(h) + \xi_1 h^3 B_3(h) + \nu_0 a_1 h^5 B_5(h) + \omega_0 \theta_0 a_1 h^7 B_7(h).$$

– *Case 2.* $\nu_0 = \omega_0 = 0$

$$\begin{aligned}\dot{x} &= -y - \theta_0 x^3 - (\eta_0 - 3\mu_0)x^2y + 3\theta_0 xy^2 - \mu_0 y^3 \\ &\quad - \varepsilon(\omega_1 - a_1)x^3 - \varepsilon(3\omega_1 + 2a_1)xy^2 + \varepsilon\nu_1 y^3 + \varepsilon^2\lambda_2 x + \varepsilon^2\xi_2 xy^2, \\ \dot{y} &= x + \mu_0 x^3 + 3\theta_0 x^2y + (\eta_0 - 3\mu_0)xy^2 - \theta_0 y^3 \\ &\quad + \varepsilon\nu_1 x^3 + \varepsilon(3\omega_1 + 2a_1)x^2y - \varepsilon a_1 y^3 + \varepsilon^2\lambda_2 y.\end{aligned}$$

The corresponding essential Melnikov function is the second one and it has the form

$$\begin{aligned}M_2(h) &= \lambda_2 h B_1(h) + \xi_2 h^3 B_3(h) + a_1 \nu_1 h^5 B_5(h) + a_1 \omega_1 h^7 B_7(h) \\ &\quad + a_1^2 (\eta_0 h^9 B_9(h) + h^{11} B_{11}(h)).\end{aligned}$$

(ii) *Generic symmetric center:* $\lambda_0 = \xi_0 = \nu_0 = \theta_0 = 0$, $a_0 \neq 0$ and $\omega_0^2 + \eta_0^2 + (4\mu_0^2 - a_0^2)^2 \neq 0$.

$$\begin{aligned}\dot{x} &= -y - (\omega_0 - a_0)x^3 - (\eta_0 - 3\mu_0)x^2y - (3\omega_0 + 2a_0)xy^2 - \mu_0 y^3 \\ &\quad + \varepsilon\lambda_1 x - \varepsilon\theta_1 x^3 + \varepsilon(3\theta_1 + \xi_1)xy^2 + \varepsilon\nu_1 y^3, \\ \dot{y} &= x + \mu_0 x^3 + (3\omega_0 + 2a_0)x^2y + (\eta_0 - 3\mu_0)xy^2 + (\omega_0 - a_0)y^3 \\ &\quad + \varepsilon\lambda_1 y + \varepsilon\nu_1 x^3 + \varepsilon 3\theta_1 x^2y - \varepsilon\theta_1 y^3.\end{aligned}$$

The corresponding essential Melnikov function is the first one and it has the form

$$\begin{aligned}M_1(h) &= \lambda_1 h B_1(h) + \xi_1 h^3 B_3(h) + \nu_1 h^5 B_5(h) + \theta_1 [\omega_0 h^7 B_7(h) + \eta_0 h^9 B_9(h) \\ &\quad + (4\mu_0^2 - a_0^2) h^{11} B_{11}(h)].\end{aligned}$$

(iii) *Generic Darboux center:* $\lambda_0 = \xi_0 = \nu_0 = \omega_0 = \eta_0 = 4(\mu_0^2 + \theta_0^2) - a_0^2 = 0$, $a_0 \neq 0$ and $\theta_0 \neq 0$.

$$\begin{aligned}\dot{x} &= -y - (\theta_0 - a_0)x^3 + 3\mu_0 x^2y - (-3\theta_0 + 2a_0)xy^2 - \mu_0 y^3 \\ &\quad + \varepsilon\lambda_1 x - \varepsilon(\omega_1 - a_1)x^3 - \varepsilon\eta_1 x^2y - \varepsilon(3\omega_1 + 2a_1 - \xi_1)xy^2 - \varepsilon\nu_1 y^3, \\ \dot{y} &= x + \mu_0 x^3 + (3\theta_0 + 2a_0)x^2y - 3\mu_0 xy^2 - (\theta_0 + a_0)y^3 \\ &\quad + \varepsilon\lambda_1 y + \varepsilon\nu_1 x^3 + \varepsilon(3\omega_1 + 2a_1)x^2y + \varepsilon\eta_1 xy^2 + \varepsilon(\omega_1 - a_1)y^3.\end{aligned}$$

The corresponding essential Melnikov function is the first one and it has the form

$$\begin{aligned}M_1(h) &= \lambda_1 h B_1(h) + \xi_1 h^3 B_3(h) + \nu_1 h^5 B_5(h) + \omega_1 h^7 B_7(h) + \eta_1 h^9 B_9(h) \\ &\quad + a_1 h^{11} B_{11}(h).\end{aligned}$$

- (iv) *Symmetric Hamiltonian center:* $\lambda_0 = \xi_0 = a_0 = \nu_0 = \theta_0 = 0$ and $\omega_0^2 + \eta_0^2 + \mu_0^2 \neq 0$.

$$\begin{aligned}\dot{x} &= -y - \omega_0 x^3 - (\eta_0 - 3\mu_0)x^2y - 3\omega_0 xy^2 - \mu_0 y^3 \\ &\quad + \varepsilon(1 - \theta_1)x^3 + \varepsilon(3\theta_1 - 2)xy^2 + \varepsilon\nu_1 y^3 + \varepsilon^2\lambda_2 x + \varepsilon^2\xi_2 xy^2, \\ \dot{y} &= x + \mu_0 x^3 + 3\omega_0 x^2y + (\eta_0 - 3\mu_0)xy^2 + \omega_0 y^3 \\ &\quad + \varepsilon\nu_1 x^3 + \varepsilon(3\theta_1 + 2)x^2y - \varepsilon(1 + \theta_1)y^3 + \varepsilon^2\lambda_2 y.\end{aligned}$$

The corresponding essential Melnikov function is the second one and it has the form

$$M_2(h) = \lambda_2 h B_1(h) + \xi_2 h^3 B_3(h) + \nu_1 h^5 B_5(h) + \omega_0 \theta_1 h^7 B_7(h).$$

- (v) *Symmetric Darboux center:* $\lambda_0 = \xi_0 = \nu_0 = \theta_0 = \omega_0 = \eta_0 = 4\mu_0^2 - a_0^2 = 0$ and $a_0 \neq 0$.

$$\begin{aligned}\dot{x} &= -y + a_0 x^3 + 3\mu_0 x^2y - 2a_0 xy^2 - \mu_0 y^3 - \varepsilon(\omega_1 - a_1 + 1)x^3 \\ &\quad - \varepsilon\eta_1 x^2y - \varepsilon(3\omega_1 + 2a_1 - 3)xy^2 + \varepsilon^2\lambda_2 x + \varepsilon^2\xi_2 xy^2 + \varepsilon^2\nu_2 y^3, \\ \dot{y} &= x + \mu_0 x^3 + 2a_0 x^2y - 3\mu_0 xy^2 - a_0 y^3 + \varepsilon(3\omega_1 + 2a_1 + 3)x^2y \\ &\quad + \varepsilon\eta_1 xy^2 + \varepsilon(\omega_1 - a_1 - 1)y^3 + \varepsilon^2\lambda_2 y + \varepsilon^2\nu_2 x^3.\end{aligned}$$

The corresponding essential Melnikov function is the second one and it has the form

$$\begin{aligned}M_2(h) &= \lambda_2 h B_1(h) + \xi_2 h^3 B_3(h) + \nu_2 h^5 B_5(h) + \omega_1 h^7 B_7(h) + \eta_1 h^9 B_9(h) \\ &\quad + a_1 h^{11} B_{11}(h).\end{aligned}$$

- (vi) *Linear center:* $\lambda_0 = \xi_0 = a_0 = \nu_0 = \theta_0 = \omega_0 = \eta_0 = \mu_0 = 0$.

$$\begin{aligned}\dot{x} &= -y - \varepsilon(\omega_1 - 1)x^3 + \varepsilon(3\theta_1 - 2)xy^2 - \varepsilon^2\eta_2 x^2y \\ &\quad - \varepsilon^3\omega_3 x^3 - 3\varepsilon^3\omega_3 xy^2 + \varepsilon^4\nu_4 y^3 + \varepsilon^5\lambda_5 x + \varepsilon^5\xi_5 xy^2, \\ \dot{y} &= x + \varepsilon(3\theta_1 + 2)x^2y - \varepsilon(\theta_1 + 1)y^3 + \varepsilon^2\eta_2 xy^2 \\ &\quad + 3\varepsilon^3\omega_3 x^2y + \varepsilon^3\omega_3 y^3 + \varepsilon^4\nu_4 x^3 + \varepsilon^5\lambda_5 y.\end{aligned}$$

The corresponding essential Melnikov function is the fifth one and it has the form

$$\begin{aligned}M_5(h) &= \lambda_5 h B_1(h) + \xi_5 h^3 B_3(h) + \nu_4 h^5 B_5(h) + \theta_1 \omega_3 h^7 B_7(h) \\ &\quad + \theta_1 \eta_2 h^9 B_9(h) + (4\theta_1^3 - \theta_1) h^{11} B_{11}(h).\end{aligned}$$

4. EXAMPLE

Consider the following system with a center at the origin

$$(12) \quad \dot{x} = -y(1 + y), \quad \dot{y} = x(1 + y),$$

having the first integral $H(x, y) = \sqrt{x^2 + y^2}$ and the corresponding inverse integrating factor $V(x, y) = (1 + y)\sqrt{x^2 + y^2}$. Its period annulus is $\mathcal{P} = \{H = h : h \in (0, 1)\}$. System (12) is in the standard Bautin form and, according to the classification of quadratic centers (given in paragraph 3.1), is a generic symmetric (reversible) center. Consider now a perturbation of system (12) by quadratic

polynomials with coefficients which are analytic in the small bifurcation parameter ε :

$$(13) \quad \dot{x} = -y(1+y) + \varepsilon p(x, y, \varepsilon), \quad \dot{y} = x(1+y) + \varepsilon q(x, y, \varepsilon).$$

As we explained in the beginning of paragraph 3.1, there exists an affine change of variables which is analytic with respect to ε that transforms system (13) in the Bautin standard form (10). This transformation is the identity for $\varepsilon = 0$, in this case, meaning that the unperturbed system (12) does not change after this transformation. Note that we have $(\lambda_{1,0}, \lambda_{2,0}, \lambda_{3,0}, \lambda_{4,0}, \lambda_{5,0}, \lambda_{6,0}) = (0, 0, 0, 1, 0, -1)$. We apply Theorem 6 (ii) and deduce that an essential perturbation of center (12) is

$$\begin{aligned} \dot{x} &= -y(1+y) + \varepsilon (\lambda_{1,1}x + (2\lambda_{2,1} + \lambda_{5,1})xy) \\ \dot{y} &= x(1+y) + \varepsilon (\lambda_{1,1}y + \lambda_{2,1}x^2 - \lambda_{2,1}y^2), \end{aligned}$$

and the essential Melnikov function is the first one and it has the form

$$M_1(h) = \lambda_{1,1}hB_1(h) + \lambda_{5,1}h^3B_3(h) + \lambda_{2,1}h^5\tilde{B}_5(h).$$

As it is proved in [3], there are at most 2 zeroes of $M_1(h)$ in \mathcal{P} .

Indeed, as it is proved in [16], when $M_1(h) \equiv 0$, the expression of the higher-order Melnikov function is analogous to $M_1(h)$. Therefore, the cyclicity of \mathcal{P} under quadratic perturbations is 2. However in [3] it is stated erroneously that the function $M_3(h)$ can have 3 zeroes. We remark that, if one uses the perturbative system considered in [3] and applies the method described in this manuscript, the same conclusion that the essential Melnikov function is the first one is accomplished.

5. ON THE FINITENESS OF THE NUMBER OF LIMIT CYCLES BIFURCATING FROM THE PERIOD ANNULUS \mathcal{P}

In this manuscript we describe a method to give an essential perturbation for a family of planar polynomials differential systems (2) which unfold a system with a period annulus \mathcal{P} corresponding to a nondegenerate center.

The existence of a essential perturbation may induce the idea that the cyclicity of any period annulus \mathcal{P} surrounding a nondegenerate center is finite. Assume that, for a particular family (3), we have that $M_{k^*}(h)$ is the essential Melnikov function where k^* is the essential order. This implies that if a particular system (3) has ℓ limit cycles which bifurcate from the orbits of \mathcal{P} , then $M_{k^*}(h)$ has at least ℓ isolated zeroes (counted with multiplicity). We recall that $M_{k^*}(h)$ is analytic in the interval $[h_0, h_1)$, where $h_0 \in \mathbb{R}$ corresponds to the inner boundary, that is the center singular point, and $h_1 \in \mathbb{R} \cup \{+\infty\}$ is the level set of the first integral $H(x, y)$ corresponding to the outer boundary of \mathcal{P} . If the number of isolated zeroes of $M_{k^*}(h)$ is finite, then the cyclicity of \mathcal{P} is finite.

Due to analyticity, any Melnikov function (and in particular the essential Melnikov function M_{k^*}) can have a countable set of zeros. Theoretically the set of zeros can be both finite and infinite. If the number of isolated zeroes of $M_{k^*}(h)$ is infinite, then they need to accumulate to h_1 (we remind that h_1 is the level value of the first integral corresponding to the outer boundary of the period annulus). The fact that this oscillatory behavior does not appear for a period annulus of a

Hamiltonian or a generic Darboux integrable system has been shown in [13]; see also the references therein.

We remark that the fact that the number of isolated zeroes of $M_{k^*}(h)$ is infinite does not contradict the finiteness of the number of limit cycles for a particular fixed system (3) which was proved by Écalle [9] and Ilyashenko [17], as we explain below. Assume, to fix ideas, that $M_{k^*}(h)$ has an infinite number of simple zeroes which we denote by ξ_n , with $n \in \mathbb{N}$. We can assume without loss of generality that $\xi_n < \xi_{n+1}$ and we have that $\lim_{n \rightarrow \infty} \xi_n = h_1$. For each ξ_n , we have by the Implicit Function Theorem (see also Theorem 1) that there exists a value $\varepsilon_n > 0$ and a function $\vartheta_n(\varepsilon)$ analytic in the interval $\varepsilon \in (-\varepsilon_n, \varepsilon_n)$ such that $d(\vartheta_n(\varepsilon); \varepsilon) \equiv 0$ for all $|\varepsilon| < \varepsilon_n$. For a fixed value $\varepsilon \in (-\varepsilon_n, \varepsilon_n) \setminus \{0\}$, the point $\vartheta_n(\varepsilon)$ corresponds to a limit cycle of the system (5) which has bifurcated from the periodic orbit corresponding to the level ξ_n . For a fixed value of ε , system (5) has a finite number of limit cycles, which implies that $\lim_{n \rightarrow \infty} \varepsilon_n \rightarrow 0$. Then, given a fixed value of $\varepsilon \neq 0$ there is a finite number of intervals in the set $\{(-\varepsilon_n, \varepsilon_n) : n \in \mathbb{N}\}$ in which ε belongs to. Thus, the functions $\vartheta_n(\varepsilon)$ only exist for this finite number of intervals and, as a consequence, there is only a finite number of limit cycles for system (5) for the considered fixed value of ε . If we take a value of ε closer to 0 we may have more limit cycles and since $\varepsilon_n > 0$ for all $n \in \mathbb{N}$ and $\lim_{n \rightarrow \infty} \varepsilon_n \rightarrow 0$, we have that given a number ℓ , there is always a value of ε close enough to 0 such that the corresponding system (5) has at least ℓ limit cycles bifurcating from the periodic orbits of \mathcal{P} . Therefore, even though for a fixed system (5) the number of limit cycles is finite, one has that the cyclicity of the period annulus \mathcal{P} is infinite.

However, it turns out that, as far as the authors know, there is no example of a Melnikov function with such an oscillatory behavior. Indeed, in all the papers known by the authors, the Melnikov function satisfies a Chebyshev property. More precisely, as we have proved in Theorem 4, $M_k(h)$ can be written as the linear combination (8) of $N + 1$ linearly independent functions $h^{2j+1}B_{2j+1}(h)$ (called Bautin functions), which are analytic for h in the whole period annulus and with $B_{2j+1}(0)$ a nonzero constant, for $j = \overline{0, N}$. It turns out, in the studied examples, that the Bautin functions are not only Chebyshev in a neighborhood of the origin but in the whole period annulus. This implies that the number of isolated zeroes (counted with multiplicity) of any linear combination of these $N + 1$ functions is at most N . We recall that given $N + 1$ analytic functions on a real interval L , they form an extended Chebyshev system (in short ET-system) on L if any nontrivial linear combination has at most N isolated zeros on L , counted with multiplicity. Some papers even conjecture such Chebyshev property for some particular systems, see [21].

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